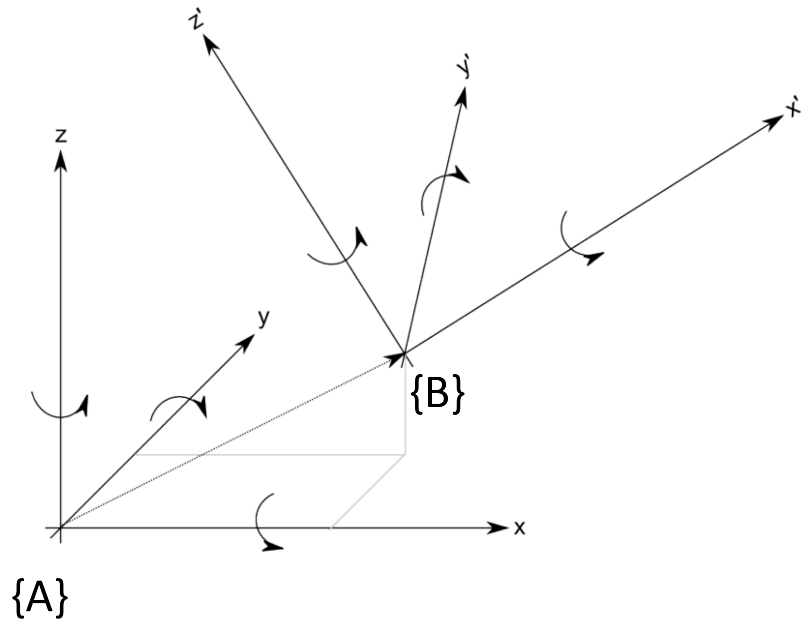
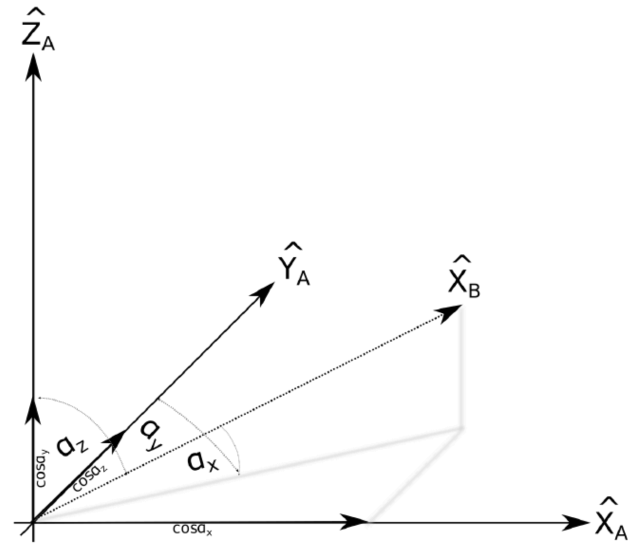
Kinematics II:
Inverse Kinematics

Chapter 3

Recall: Rotations and Translations





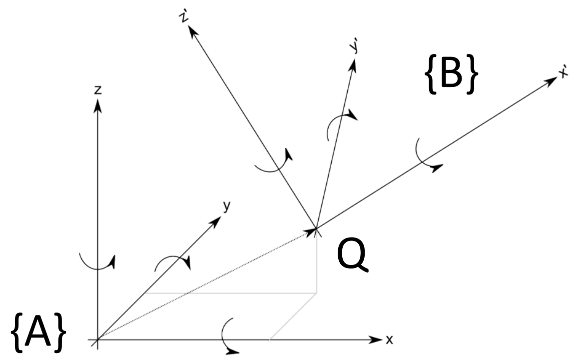
Rotation Matrix:

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & -sin\gamma \\ 0 & sin\gamma & cos\gamma \end{bmatrix}
$$

Each column is a vector of coordinate system B expressed in A

Recall: Transformation Arithmetic

Expressing a point Q in {B} in the coordinates of {A}



$$^{A}Q = {}^{A}_{B}R\,{}^{B}Q + {}^{A}P$$

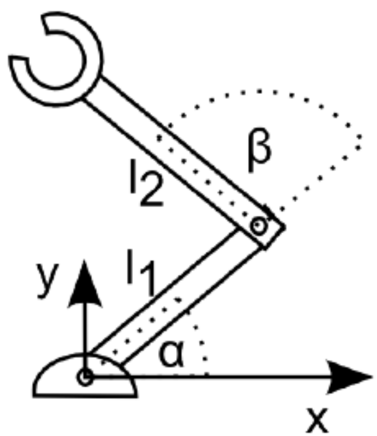
Better: Homogeneous Transform

$$\begin{bmatrix} {}^{A}Q \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}Q \\ 1 \end{bmatrix}$$

$^{A}_{B}T$

$$^{A}Q = {}^{A}_{B}T\,{}^{B}Q$$

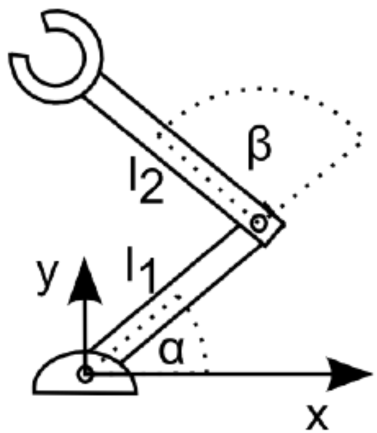
Today: Inverse Kinematics



What angles do I need to set my joints to reach a desired pose?

$$
\begin{aligned}
x &= \cos(\alpha+\beta)l_2 + \cos\alpha l_1 \\
y &= \sin(\alpha+\beta)l_2 + \sin\alpha l_1
\end{aligned}
$$

Inverse Kinematics of a 2-link Arm



$$x_1 \quad = \quad \cos\alpha l_1$$

↓

$$\left[\cos^{-1}\frac{x_1}{l_1}, -\cos^{-1}\frac{x_1}{l_1}\right]$$

$$\alpha \to \cos^{-1}\left(\frac{x^2y + y^3 - \sqrt{4x^4 - x^6 + 4x^2y^2 - 2x^4y^2 - x^2y^4}}{2(x^2+y^2)}\right)$$

$$\beta \to -\cos^{-1}\left(1/2(-2 + x^2 + y^2)\right)$$

Easier ways to solve the IK problem

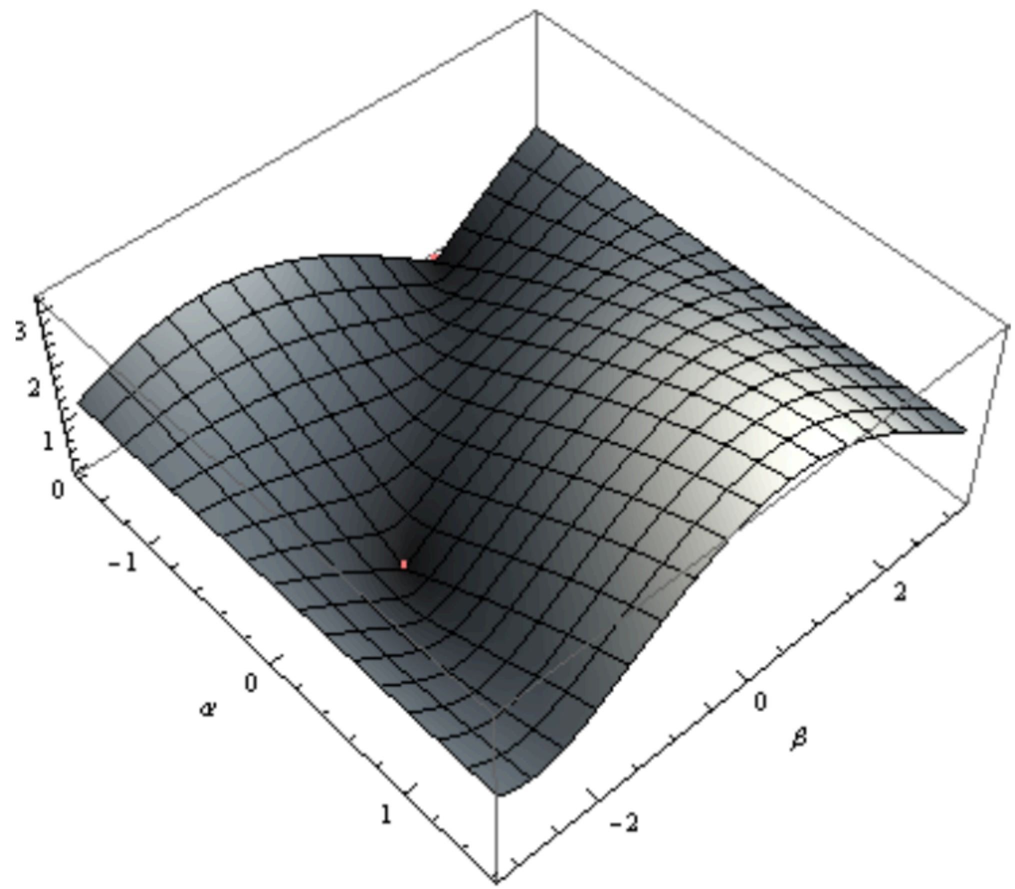
$$\begin{pmatrix} cos_{\alpha\beta} & -sin_{\alpha\beta} & 0 & \cos_{\alpha\beta} l_2 + \cos\alpha l_1 \\ sin_{\alpha\beta} & cos_{\alpha\beta} & 0 & \sin_{\alpha\beta} l_2 + \sin\alpha l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} cos\phi & -sin\phi & 0 & x \\ sin\phi & cos\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation + Translation

Desired x/y/orientation

$$\begin{aligned} \phi &= \alpha + \beta \\ \cos\alpha &= \frac{\cos_{\alpha\beta} l_2 - x}{l_1} = \frac{\cos\phi l_2 - x}{l_1} \\ \sin\alpha &= \frac{\sin_{\alpha\beta} l_2 - x}{l_1} = \frac{\sin\phi l_2 - x}{l_1} \end{aligned}$$

Easier ways to solve the IK problems



$$f_{x,y}(\alpha, \beta) = \sqrt{(\sin(\alpha + \beta) + \sin(\alpha) - y)^2 + (\cos(\alpha + \beta) + \cos(a) - x)^2}$$

Inverse Kinematics of Mobile Robots

$$\begin{pmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{\dot{\phi}_r r}{d} - \frac{\dot{\phi}_l r}{d} \end{pmatrix}$$

$$\dot{\xi_I} = T(\theta)\dot{\xi_R}$$

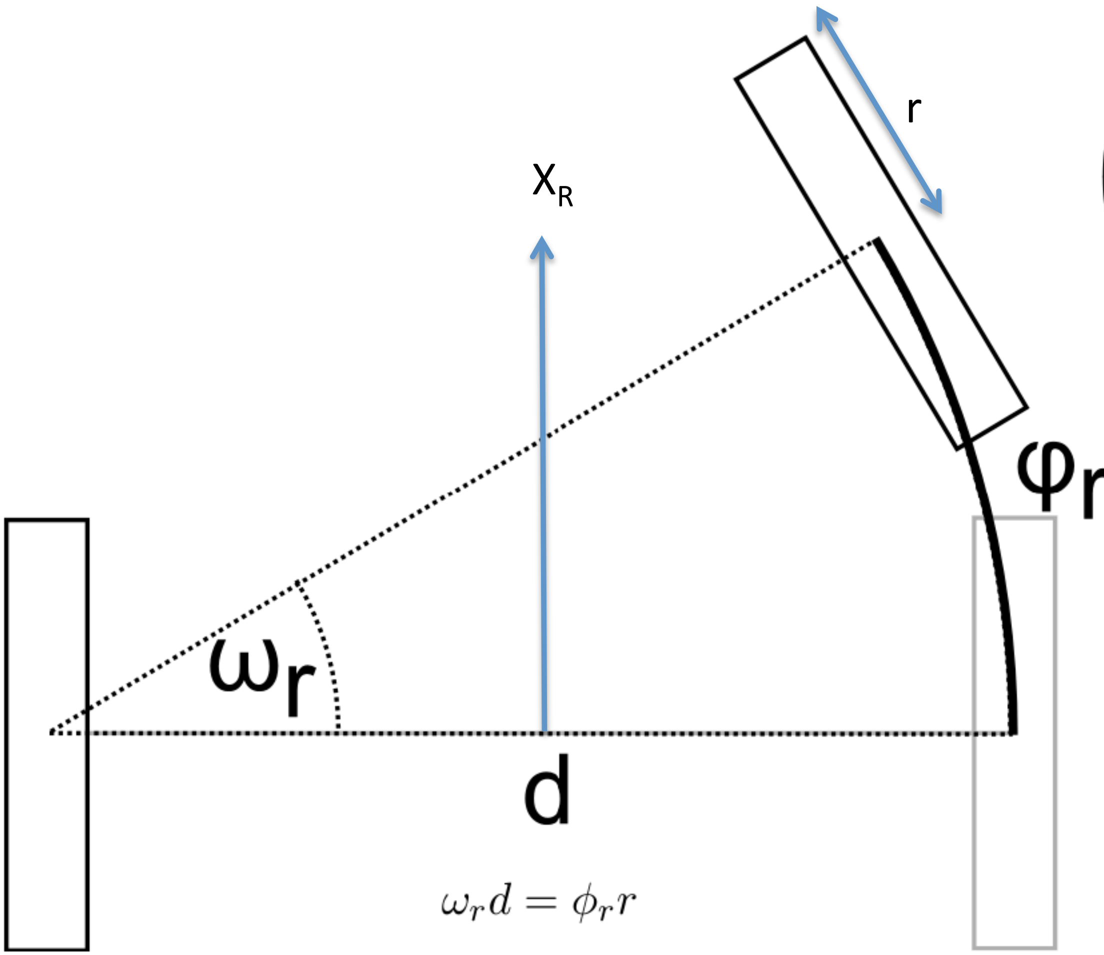
$$T^{-1}(\theta)\dot{\xi_I} = T^{-1}(\theta)T(\theta)\dot{\xi_R}$$

$$T^{-1} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\dot{\xi_R} = T^{-1}(\theta)\dot{\xi_I}$$

Recall: Kinematics of a Mobile Robot

$$\begin{pmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{r\dot{\phi_l}}{2} + \frac{r\dot{\phi_r}}{2} \\ 0 \\ \frac{\dot{\phi_r} r}{d} - \frac{\dot{\phi_l} r}{d} \end{pmatrix}$$

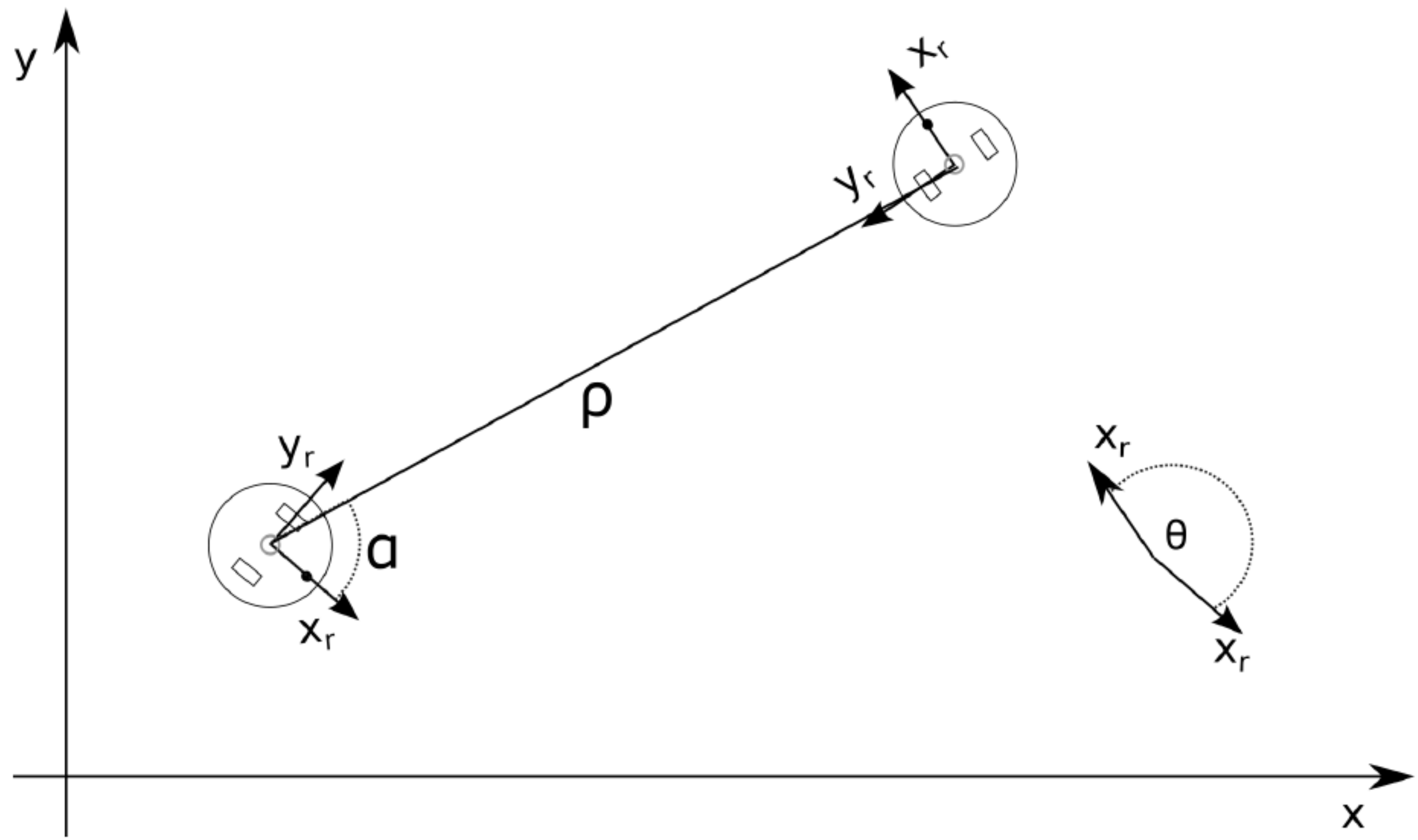


$$\omega_r d = \phi_r r$$

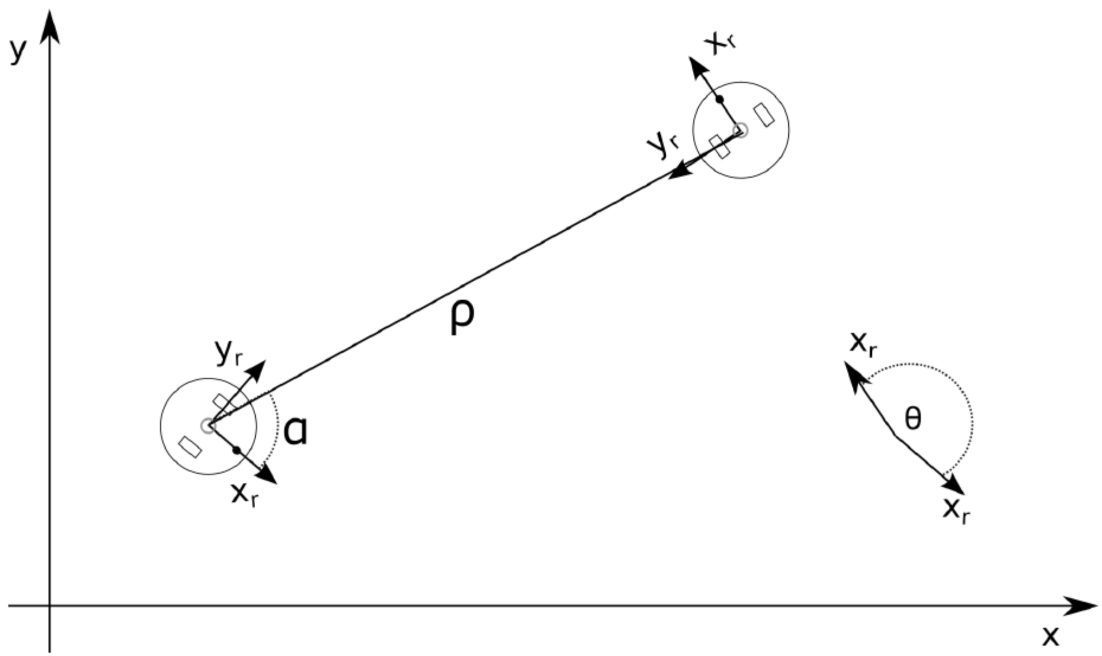
Inverse Kinematics of Mobile Robots

$$\begin{pmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{\dot{\phi}_r r}{d} - \frac{\dot{\phi}_l r}{d} \end{pmatrix} \longrightarrow \begin{aligned} \dot{\phi}_l &= (2\dot{x}_R/r - \dot{\theta}d)/2 \\ \dot{\phi}_r &= (2\dot{x}_R/r + \dot{\theta}d)/2 \end{aligned}$$

Position control using feedback control



Position control using feedback control



$$\rho = \sqrt{(x_r - x_g)^2 + (y_r - y_g)^2}$$

$$\alpha = \theta_r - \tan^{-1} \frac{y_r - y_g}{x_r - x_g}$$

$$\eta = \theta_g - \theta_r$$

$$\begin{aligned} \dot{x} &= p_1 \rho \\ \dot{\theta} &= p_2 \alpha + p_3 \eta \end{aligned}$$

Summary: Inverse Kinematics of a Mobile Robot

- Calculate suitable velocities that drive the robot toward your goal

$$\begin{aligned} \dot{x} &= p_1\rho \\ \dot{\theta} &= p_2\alpha + p_3\eta \end{aligned}$$

- Calculate the necessary wheel-speed

$$\begin{aligned} \dot{\phi}_l &= (2\dot{x}_R/r - \dot{\theta}d)/2 \\ \dot{\phi}_r &= (2\dot{x}_R/r + \dot{\theta}d)/2 \end{aligned}$$

- Problem
  - How to deal with obstacles?
  - How to find short(est) paths?
- Chapter 4: Path Planning