Uncertainty and Error Propagation

Chapter 8
Last week: Features

• Features are a smart way to reduce data coming from sensors
• Features are task-relevant high-level information
• Least-squares gives optimal solutions
• RANSAC deals with outliers
• Feature extraction is an optimization problem with a probabilistic outcome

\[ S_{r,\alpha} = \sum_i d_i^2 = \sum_i (\rho_i \cos(\theta_i - \alpha) - r)^2 \]
Topics in this class so far...

**Perception**
- Forward Kinematics (Odometry)
- Sensors for position, velocity and acceleration
- Feature extraction

**Reasoning**
- Inverse Kinematics
- Path planning

**Probabilistic**

**Deterministic**
Uncertainty: Odometry

• Sources
  – Wheel-slip
  – Timing problems (during integration)
  – Neglecting true dynamics
Uncertainty: Ultrasound

- ToF measurement error
- Multi-path reflections
- Material properties
Uncertainty: Feature extraction

- Sensor noise propagates into least-squares solution
- Feature extraction itself probabilistic

One of many possible solutions returned by RANSAC
Today

• How to formally describe uncertainty
  – Random variables, Probability Density Functions
• How to combine different random processes
  – Convolution
• How uncertainty propagates from measurement to feature
  – Error propagation
Random Variable

- Eyes on a die
- Sum of eyes on two dice
- Distance to the wall
- Position on a map
- Reading from a light sensor
- In the kitchen (yes/no)
- ...

Samples of a random variable are called variates
Example: Throwing dice

- One die: 4,5,1,3,2,4,5,2,2,6,7,8,...
- Each variate has a probability of 1/6
- Uniform distribution
- What’s the distribution of the sum of two dice?
The distribution of the sum of two random variables is the convolution of their distributions.

Example

The distribution of the sum of two random variables is the convolution of their distributions.
The Gaussian ("Normal") Distribution

\[ f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

\[ \mu = \int_{-\infty}^{\infty} x f(x) \, dx \]

\[ \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \]
2D Gaussian Distribution

How does uncertainty propagate?
Error Propagation Law

• Sometimes random variates are combinations of others
• Example: x, y and theta result from wheel slip (two random variates)
• If the PDFs are Gaussian, their variances add up
• Intuition: weigh each component with their variance
Error propagation

• Random variable $Y$ is a function of random variable $X$
  \[ y = f(x) \]

• New variance

• Weighed by the gradient with respect to $X$

• Measure of how important a change in $X$ is to $Y$

• Multi-input, Multi-output leads to covariance matrices
  \[ \Sigma^Y = J \Sigma^X J^T \]
Example: Odometry

1. Forward Kinematics
   (maps wheel slip to pose)

   \[
   f(x, y, \theta, \Delta s_r, \Delta s_l) = \begin{bmatrix} x, y, \theta \end{bmatrix}^T + \begin{bmatrix} \Delta x & \Delta y & \Delta \theta \end{bmatrix}^T
   \]

   \[
   \Delta x = \Delta s \cos(\theta + \Delta \theta/2) \\
   \Delta y = \Delta s \sin(\theta + \Delta \theta/2) \\
   \Delta \theta = \frac{\Delta s_r - \Delta s_l}{2} \\
   \Delta s = \frac{\Delta s_r + \Delta s_l}{2}
   \]

2. Error update

   \[
   \Sigma_p' = \nabla_p \Sigma_p \nabla_p f^T + \nabla_{\Delta r,l} f \Sigma \nabla \Delta_{r,l} f^T
   \]

   Component from Additional
   Motion wheel-slip
Example: Odometry

3. Partial derivatives of kinematics with respect to pose

\[ \nabla_p f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta s\sin(\theta + \Delta \theta/2) \\ 0 & 1 & \Delta s\cos(\theta + \Delta \theta/2) \\ 0 & 0 & 1 \end{bmatrix} \]

4. Partial derivatives of kinematics with respect to wheel-slip

\[ \nabla_{\Delta_r, l} f = \begin{bmatrix} \frac{1}{2} \cos(\theta + \Delta \theta/2) & \frac{1}{2} \cos(\theta + \Delta \theta/2) \\ \frac{1}{2} \sin(\theta + \Delta \theta/2) & \frac{1}{2} \cos(\theta + \Delta \theta/2) \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \]

\[
\Sigma'_p = \nabla_p f^T \Sigma_p \nabla_p f + \nabla_{\Delta_r, l} f^T \Sigma \nabla_{\Delta_r, l} f^T
\]

\[
\Sigma_\Delta = \begin{bmatrix} \frac{k_r}{2} |\Delta s_r| & 0 \\ 0 & \frac{k_l}{2} |\Delta s_l| \end{bmatrix}
\]

Wheel-slip covariance matrix
Demo: Odometry error
Summary

• Most variables describing a robot’s state are *random variables*
• Variates of a random variable are drawn from *Probability Density Functions (PDF)*
• A common, because convenient, PDF is the Gaussian Distribution defined by its mean and variance
• For Gaussians, variances add up and are weighed by the impact they have on the combined random variable (“Error Propagation Law”)