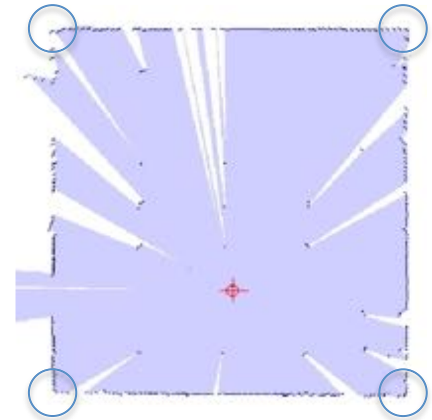
Uncertainty and Error Propagation

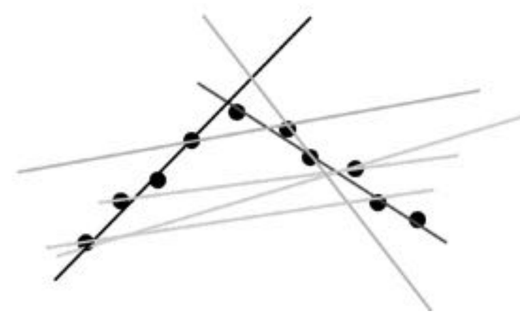
Chapter 8

Last week: Features

- Features are a smart way to reduce data coming from sensors
- Features are task-relevant high-level information
- Least-squares gives optimal solutions
- RANSAC deals with outliers
- Feature extraction is an optimization problem with a probabilistic outcome



$$S_{r,\alpha} = \sum_i d_i^2 = \sum_i (\rho_i \cos(\theta_i - \alpha) - r)^2$$



Topics in this class so far…

Perception

- Forward Kinematics (Odometry)
- Sensors for position, velocity and acceleration
- Feature extraction

Probabilistic

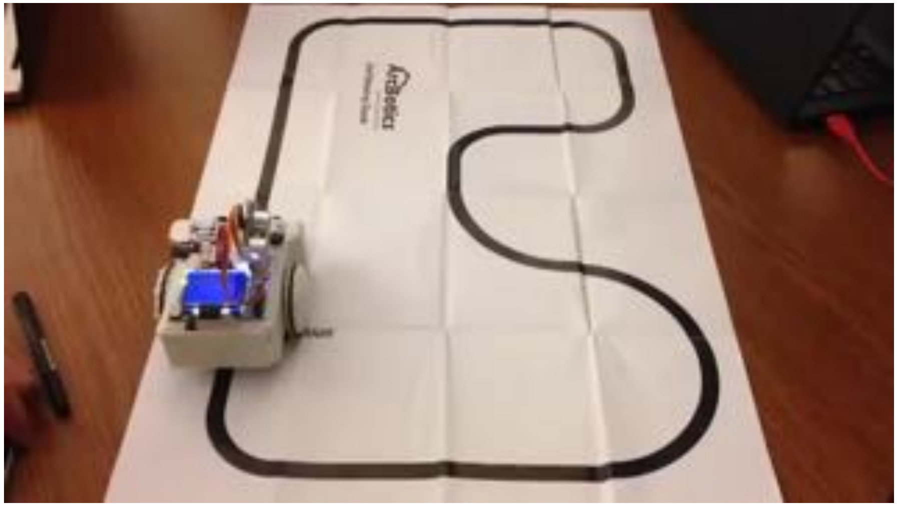
Reasoning

- Inverse Kinematics
- Path planning

Deterministic

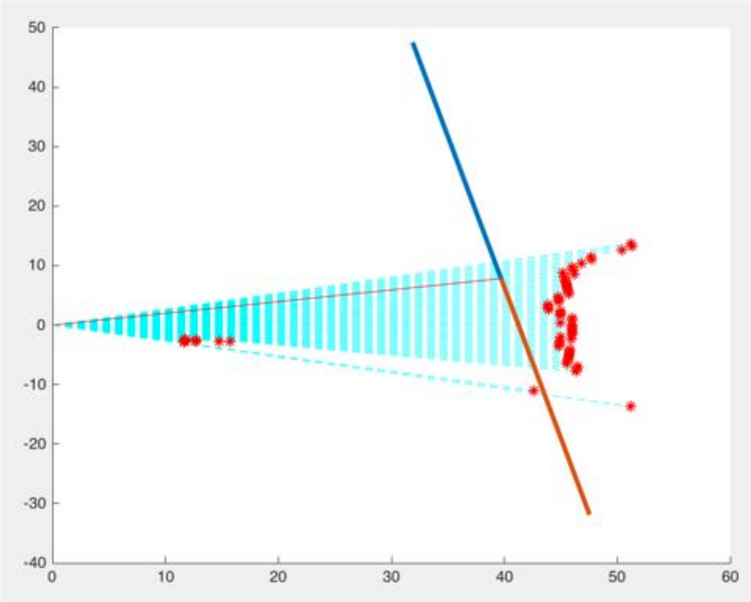
Uncertainty: Odometry

- Sources
  - Wheel-slip
  - Timing problems (during integration)
  - Neglecting true dynamics



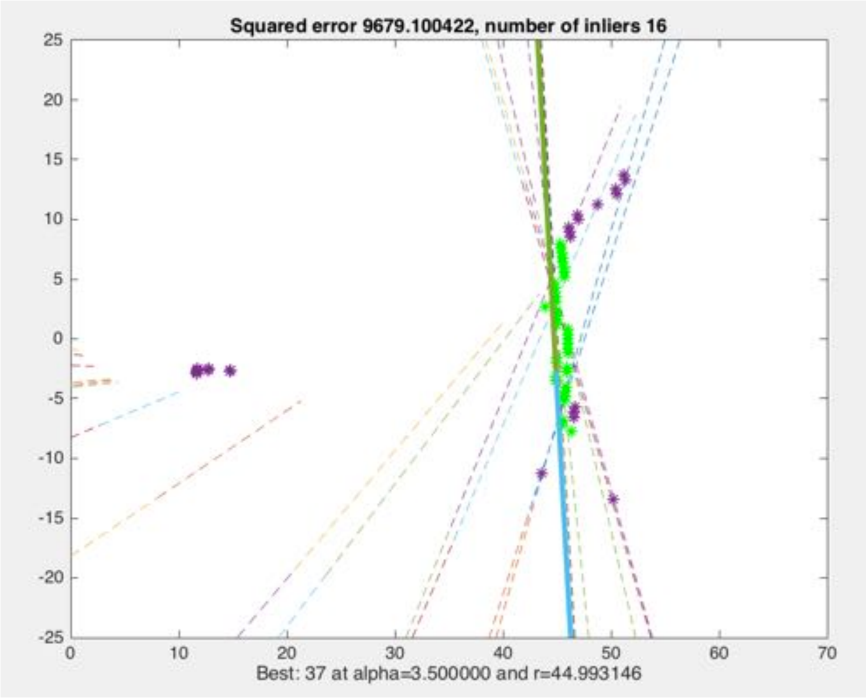
Uncertainty: Ultrasound

- ToF measurement error
- Multi-path reflections
- Material properties



Uncertainty: Feature extraction

- Sensor noise propagates into least-squares solution
- Feature extraction itself probabilistic



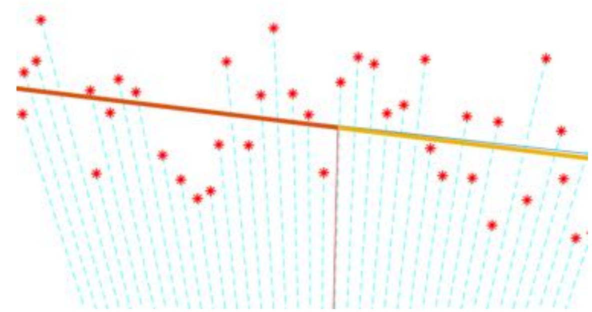
One of many possible solutions returned by RANSAC

Today

- How to formally describe uncertainty
  - Random variables, Probability Density Functions
- How to combine different random processes
  - Convolution
- How uncertainty propagates from measurement to feature
  - Error propagation

Random Variable

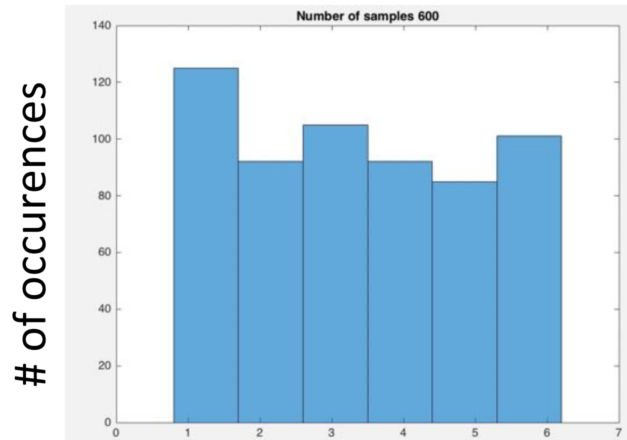
- Eyes on a die
- Sum of eyes on two dice
- Distance to the wall
- Position on a map
- Reading from a light sensor
- In the kitchen (yes/no)
- …

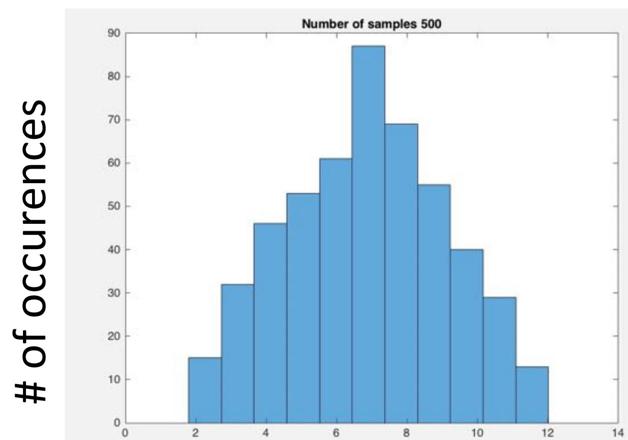


Samples of a random variable are called variates

Example: Throwing dice

- One die: 4,5,1,3,2,4,5,2,2,6,7,8,…
- Each variate has a probability of 1/6
- <u>Uniform distribution</u>
- What's the distribution of the sum of two dice?





Sum of two probability distributions

$$Z = X + Y$$
$$X = k$$
$$Y = z - k$$

Example

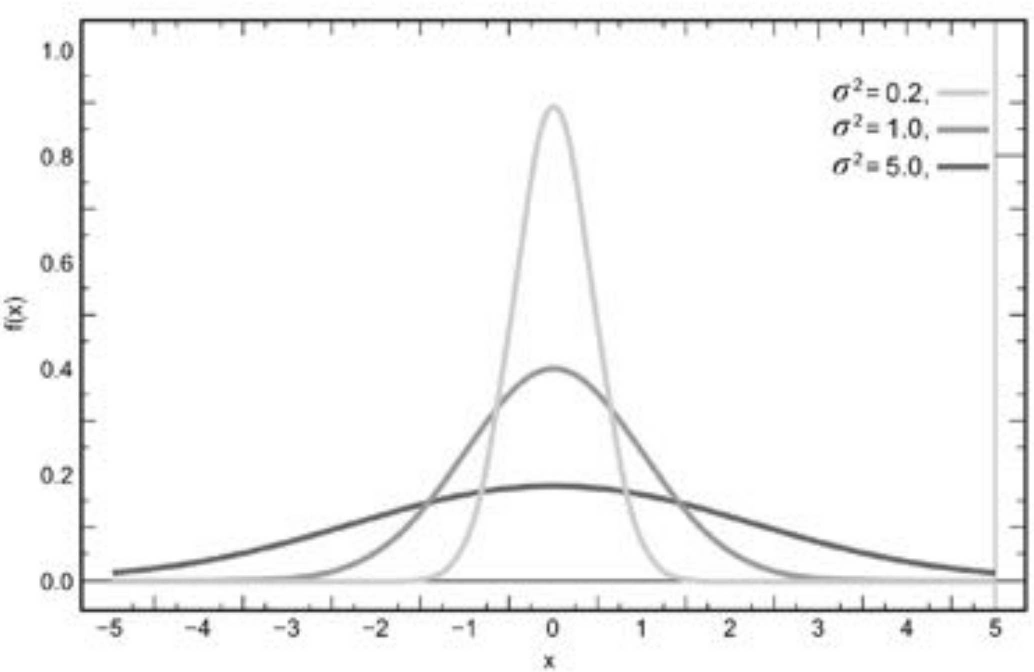
$$\begin{aligned}
&2 : 1+1 && \rightarrow \tfrac{1}{6}\tfrac{1}{6} \\
&3 : 1+2, 2+1 && \rightarrow 2\tfrac{1}{6}\tfrac{1}{6} \\
&4 : 1+3, 2+2, 3+1 && \rightarrow 3\tfrac{1}{6}\tfrac{1}{6} \\
&5 : 1+4, 2+3, 3+2, 4+1 && \rightarrow 4\tfrac{1}{6}\tfrac{1}{6} \\
&6 : 1+5, 2+4, 3+3, 4+2, 5+1 && \rightarrow 5\tfrac{1}{6}\tfrac{1}{6} \\
&7 : 1+6, 2+5, 3+4, 4+3, 5+2, 6+1 && \rightarrow 6\tfrac{1}{6}\tfrac{1}{6} \\
&8 : 2+6, 3+5, 4+4, 5+3, 6+2 && \rightarrow 5\tfrac{1}{6}\tfrac{1}{6} \\
&9 : 3+6, 4+5, 5+4, 6+3 && \rightarrow 4\tfrac{1}{6}\tfrac{1}{6} \\
&10 : 4+6, 5+5, 6+4 && \rightarrow 3\tfrac{1}{6}\tfrac{1}{6} \\
&11 : 5+6, 6+5 && \rightarrow 2\tfrac{1}{6}\tfrac{1}{6} \\
&12 : 6+6 && \rightarrow \tfrac{1}{6}\tfrac{1}{6}
\end{aligned}$$

$$P(Z = z) = \sum_{k=-\infty}^{\infty} P(X = k) P(Y = z - k)$$

$$P(Z) = P(X) \star P(Y)$$

The distribution of the sum of two random variables is the convolution of their distributions.

The Gaussian ("Normal") Distribution

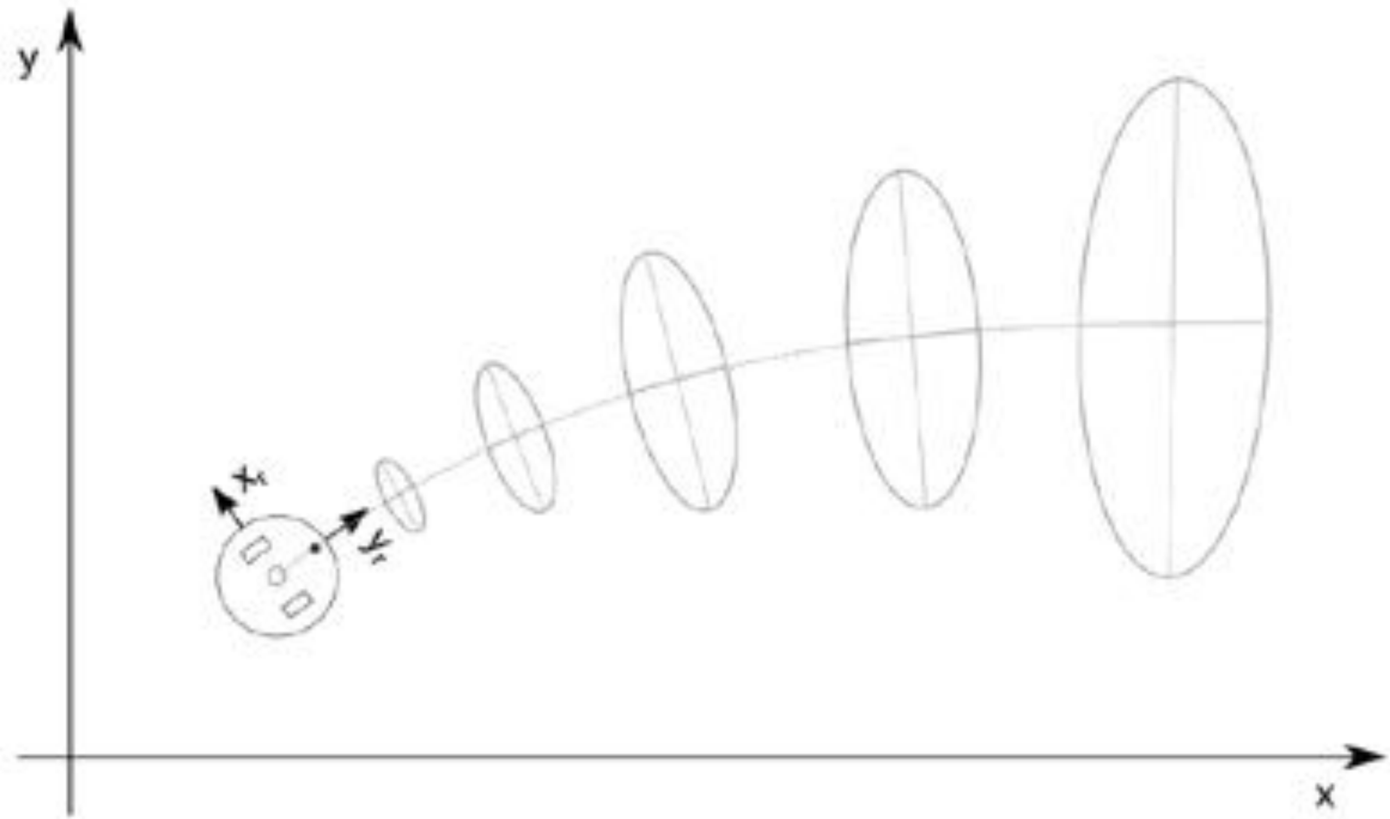


$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

2D Gaussian Distribution



How does uncertainty propagate?

Error Propagation Law

- Sometimes random variates are combinations of others
- Example: x, y and theta result from wheel slip (two random variates)
- If the PDFs are Gaussian, their variances add up
- Intuition: weigh each component with their variance

Error propagation

- Random variable Y is a function of random variable X $y = f(x)$
- New variance $\sigma_y^2 = \frac{\partial df}{\partial x}^2 \sigma_x^2$
- Weighed by the gradient with respect to X
- Measure of how important a change in X is to Y
- Multi-input, Multi-output leads to covariance matrices $\Sigma^Y = J\Sigma^X J^T$

Example: Odometry

1. Forward Kinematics
   (maps wheel slip to pose)

$$\begin{aligned} \Delta x &= \Delta s cos(\theta + \Delta\theta/2) \\ \Delta y &= \Delta s sin(\theta + \Delta\theta/2) \\ \Delta\theta &= \frac{\Delta s_r - \Delta s_l}{2} \end{aligned} \qquad \Delta s = \frac{\Delta s_r + \Delta s_l}{2}$$

$$f(x, y, \theta, \Delta s_r, \Delta s_l) = [x, y, \theta]^T + [\Delta x \quad \Delta y \quad \Delta\theta]^T$$

2. Error update

$$\Sigma_{p'} = \nabla_p f \Sigma_p \nabla_p f^t + \nabla_{\Delta_{r,l}} f \Sigma_\Delta \nabla_{\Delta_{r,l}} f^T$$

Component from Motion

Additional wheel-slip

Example: Odometry

$$\begin{aligned} \Delta x &= \Delta s cos(\theta + \Delta\theta/2) \\ \Delta y &= \Delta s sin(\theta + \Delta\theta/2) \\ \Delta\theta &= \frac{\Delta s_r - \Delta s_l}{2} \end{aligned}$$

3. Partial derivatives of kinematics with respect to pose

$$\nabla_p f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta s sin(\theta + \Delta\theta/2) \\ 0 & 1 & \Delta s cos(\theta + \Delta\theta/2) \\ 0 & 0 & 1 \end{bmatrix}$$

4.Partial derivates of kinematics with respect to wheel-slip

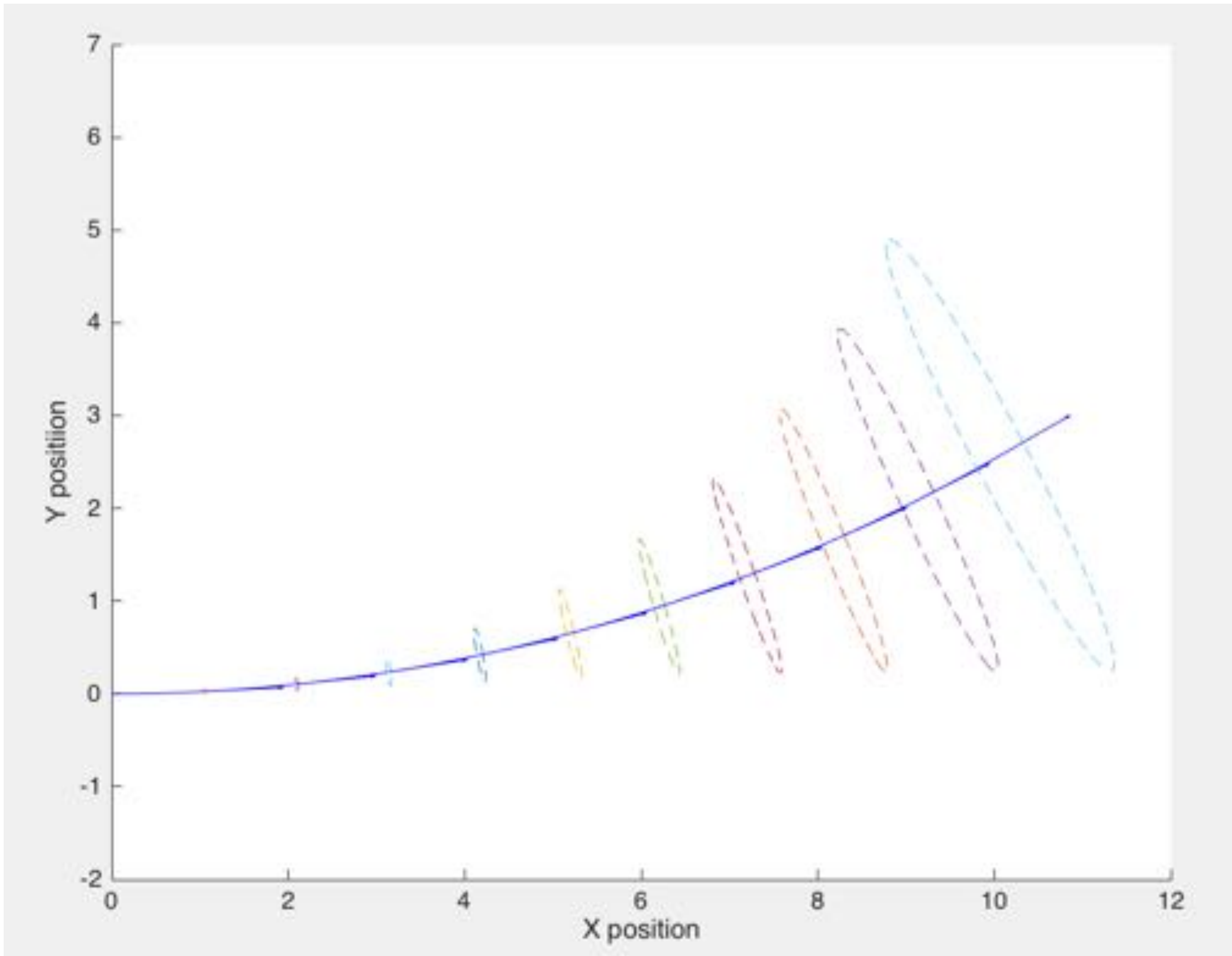
$$\nabla_{\Delta_{r,l}} f = \begin{bmatrix} \frac{1}{2}\cos(\theta + \Delta\theta/2) & \frac{1}{2}\cos(\theta + \Delta\theta/2) \\ \frac{1}{2}\sin(\theta + \Delta\theta/2) & \frac{1}{2}\cos(\theta + \Delta\theta/2) \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\boxed{\Sigma_{p'} = \nabla_p f \Sigma_p \nabla_p f^t + \nabla_{\Delta_{r,l}} f \Sigma_\Delta \nabla_{\Delta_{r,l}} f^T}$$

$$\Sigma_\Delta = \begin{bmatrix} k_r|\Delta s_r| & 0 \\ 0 & k_l|\Delta s_l| \end{bmatrix}$$

Wheel-slip covariance matrix

Demo: Odometry error



Summary

- Most variables describing a robot's state are *random variables*
- Variates of a random variable are drawn from *Probability Density Functions (PDF)*
- A common, because convenient, PDF is the Gaussian Distribution defined by its mean and variance
- For Gaussians, variances add up and are weighed by the impact they have on the combined random variable ("Error Propagation Law")